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Integrals of Rational Symmetric Functions, Two–Matrix Models and Biorthogonal Polynomials[1]

J. Harnad[†‡2] and A. Yu. Orlov[⋆3]

[†] *Centre de recherches mathématiques, Université de Montréal*
C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7

[‡] *Department of Mathematics and Statistics, Concordia University*
7141 Sherbrooke W., Montréal, Québec, Canada H4B 1R6

[⋆] *Nonlinear Wave Processes Laboratory,*
Oceanology Institute, 36 Nakhimovskii Prospect
Moscow 117851, Russia

Abstract

We give a new method for the evaluation of a class of integrals of rational symmetric functions in N pairs of variables $\{(x_a, y_a)\}_{a=1,\dots N}$ arising in coupled matrix models, valid for a broad class of two-variable measures. The result is expressed as the determinant of a matrix whose entries consist of the associated biorthogonal polynomials, their Hilbert transforms, evaluated at the zeros and poles of the integrand, and bilinear expressions in these. The method is elementary and direct, using only standard determinantal identities, partial fraction expansions and the property of biorthogonality. The corresponding result for one-matrix models and integrals of rational symmetric functions in N variables $\{x_a\}_{a=1,\dots N}$ is also rederived in a simple way using this method.

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[2]harnad@crm.umontreal.ca

[3]orlovs@wave.sio.rssi.ru

1 Integrals of symmetric rational functions

Let $d\mu(x,y)$ be a measure (in general, complex), supported on a finite set of products of curves in the complex x and y planes, for which the semi-infinite matrix of bi-moments is finite:

$$B_{jk} := \int d\mu(x,y) x^j y^k < \infty, \quad 0, \quad \forall j,k \in \mathbf{N}. \tag{1.1}$$

The integrals are understood to be evaluated on a specified linear combination of products of the support curves. Assuming that, for all $N \geq 1$, the $N \times N$ submatrix $(B_{jk})_{0\leq j,k,\leq N-1}$ is nonsingular, the Gram-Schmidt process may be used to construct an infinite sequence of pairs of biorthogonal polynomials $\{P_j(x), S_j(y)\}_{j=0\ldots\infty}$, unique up to signs, satisfying

$$\int d\mu(x,y) P_j(x) S_k(y) = \delta_{jk}, \tag{1.2}$$

and normalized to have leading coefficients that are equal:

$$P_j(x) = \frac{x^j}{\sqrt{h_j}} + O(x^{j-1}), \qquad S_j(x) = \frac{y^j}{\sqrt{h_j}} + O(y^{j-1}). \tag{1.3}$$

We will also assume that the Hilbert transforms of these biorthogonal polynomials,

$$\tilde{P}_n(\mu) := \int d\mu(x,y) \frac{P_m(x)}{\mu - y}, \quad \tilde{S}_n(\eta) := \int d\mu(x,y) \frac{S_m(x)}{\eta - x}, \tag{1.4}$$

exist for all $n \in \mathbf{N}$.

For $N \geq 1$, let

$$\mathbf{Z}_N^{(2)} := \int d\mu(x_1,y_1) \ldots \int d\mu(x_N,y_N) \Delta_N(x) \Delta(y) = N! \prod_{n=0}^{N} h_n \tag{1.5}$$

where

$$\Delta_N(x) := \prod_{i>j}^{N} (x_i - x_j), \quad \Delta_N(y) := \prod_{i>j}^{N} (y_i - y_j) \tag{1.6}$$

are Vandermonde determinants. Such integrals are of particular interest in two-matrix models [3, 4, 7, 9, 13, 11] and may be interpreted as the reduction to the space of eigenvalues of the integral defining the partition function on an ensemble of pairs of $N \times N$ matrices having a probability measure that is invariant with respect to conjugation by unitary matrices, and admitting a Harish-Chandra-Itzykson-Zuber [13] type reduction

to the $2N$ dimensional space of eigenvalues. The joint probability distribution (in the case of a real Borel measure) is given, after normalization, by the integrand of (1.5) with eigenvalues $(x_1,\dots x_N),(y_1,\dots y_N)$ having values on the support curves of the measure $d\mu(x,y)$.

Now choose four sets of complex constants $\{\xi_\alpha,\zeta_\beta,\eta_j,\mu_k\}_{\substack{\alpha=1,\dots L_1,\beta=1,\dots L_2\\ j=1\dots M_1,k=1\dots M_2}}$, distinct within each of the four groups, with

$$N+L_1-M_1\geq N+L_2-M_2\geq 0 \tag{1.7}$$

and such that the η's and μ's are not on the support curves of the bimeasure $d\mu(x,y)$ in the x and y planes, respectively. (There is no loss of generality in assuming the first inequality in (1.7), since the results for the reversed case can be read off by symmetry in the two sets of variables.) The main result of this work is a new derivation of the following expression for the integral of a symmetric, rational function in two sets of N variables $\{x_a,y_a\}_{a=1,\dots N}$, having simple zeros at the points $\{\xi_\alpha\}_{\alpha=1\dots L_1}$, $\{\zeta_\beta\}_{\beta=1\dots L_2}$ and simple poles at $\{\eta_j\}_{j=1\dots M_1}$, $\{\mu_k\}_{k=1\dots M_2}$ in terms of the biorthogonal polynomials $\{P_n,S_n\}$, evaluated at the points $\{\xi_\alpha\}_{\alpha=1\dots L_1}$, $\{\zeta_\beta\}_{\beta=1\dots L_2}$, and bilinear combinations of these, together with the Hilbert transforms $\{\tilde P_n,\tilde S_n\}$ evaluated at the points $\{\mu_k\}_{k=1\dots M_2}$, $\{\eta_j\}_{j=1\dots M_1}$.

$$\begin{aligned}
\mathbf{I}_N^{(2)} &:= \frac{1}{\mathbf{Z}_N^{(2)}}\int d\mu(x_1,y_1)\dots\int d\mu(x_N,y_N)\Delta_N(x)\Delta_N(y)\\
&\qquad\times\prod_{a=1}^N\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha-x_a)\prod_{\beta=1}^{L_2}(\zeta_\beta-y_a)}{\prod_{j=1}^{M_1}(\eta_j-x_a)\prod_{k=1}^{M_2}(\mu_k-y_a)} &(1.8)\\
&= \epsilon(L_1,L_2,M_2,M_2)\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}\\
&\quad\times\frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\,\det G, &(1.9)
\end{aligned}$$

where

$$\epsilon(L_1,L_2,M_2,M_2):=(-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}(-1)^{L_1M_2} \tag{1.10}$$

and G is the $(L_1+M_2)\times(L_1+M_2)$ matrix

$$G=\begin{pmatrix}\overset{N+L_2-M_2}{K_{11}}(\xi_\alpha,\eta_j) & \overset{N+L_2-M_2}{K_{12}}(\xi_\alpha,\zeta_\beta) & P_{N+L_2-M_2}(\xi_\alpha) & \dots & P_{N+L_1-M_1-1}(\xi_\alpha)\\ \overset{N+L_2-M_2}{K_{21}}(\mu_k,\eta_j) & \overset{N+L_2-M_2}{K_{22}}(\mu_k,\zeta_\beta) & \tilde P_{N+L_2-M_2}(\mu_k) & \dots & \tilde P_{N+L_1-M_1-1}(\mu_k)\end{pmatrix} \tag{1.11}$$

with

$$\overset{J}{K}_{12}(\xi,\zeta) := \sum_{n=0}^{J-1} P_n(\xi) S_n(\zeta) \tag{1.12}$$

$$\overset{J}{K}_{11}(\xi,\eta) := \sum_{n=0}^{J-1} P_n(\xi) \tilde{S}_n(\eta) + \frac{1}{\xi-\eta} \tag{1.13}$$

$$\overset{J}{K}_{22}(\mu,\zeta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu) S_n(\zeta) + \frac{1}{\zeta-\mu} \tag{1.14}$$

$$\overset{J}{K}_{21}(\mu,\eta) := \sum_{n=0}^{J-1} \tilde{P}_n(\mu) \tilde{S}_n(\eta) - H(\mu,\eta) \tag{1.15}$$

$$H(\mu,\eta) := \int \frac{d\mu(x,y)}{(\eta-x)(\mu-y)}. \tag{1.16}$$

Similar expressions for the cases when one or both of the inequalities (1.7) is reversed will also be derived. (See Section 3.)

Within the setting of two-matrix models, the integral (1.9) may be interpreted as the expectation value of the product of L_1 evaluations of the characteristic polynomial of the first matrix and L_2 of that of the second matrix, divided by similar products of M_1 and M_2 further evaluations of the respective characteristic polynomials. Integrals of this type were computed in the context of complex matrix models, where the variables (x_a, y_a) are replaced by pairs $(z_a, \bar{z}_a)$ of complex conjugate values, in [1] for the special case when $M_1 = M_2 = 0$, and, more generally, in [2], for all values of (L_1, L_2, M_1, M_2). The method of derivation used by these authors was based essentially upon recursive arguments, and is rather lengthy compared with the "direct method" that we present here. Other cases of the integral (1.9), in which the condition (1.7) does not hold, give rise to analogous determinantal expressions. These too will be derived in Section 3, using the same methods as those leading to (1.9).

Remark 1.1 Note that the multiplicative factor in front of the $\det g$ term in (1.9) is just a product of the inverse determinants that enter into rational interpolation formulae; e.g., if $L_1 \geq M_1$, $L_2 \geq M_2$, then

$$\frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} = \frac{1}{\det G_1 \det G_2} \tag{1.17}$$

where

$$G_1 := \begin{pmatrix} \frac{1}{\xi_\alpha-\eta_j} & \xi_\alpha^b \end{pmatrix}_{\substack{1\leq\alpha\leq L_1,\ 1\leq j\leq M_1 \\ b\leq L_1-M_1-1}}, \quad G_2 := \begin{pmatrix} \frac{1}{\zeta_\beta-\mu_k} & \zeta_\beta^c \end{pmatrix}_{\substack{1\leq\beta\leq L_2,\ 1\leq j\leq M_2 \\ c\leq L_2-M_2-1}}, \tag{1.18}$$

Remark 1.2 Expressions for the biorthogonal polynomials $\{P_n^{(\xi,\zeta,\eta,\mu)}(x), S_n^{(\xi,\zeta,\eta,\mu)}(y)\}_{n\in\mathbf{N}}$ with respect to the modified measure

$$d\mu_{\xi,\zeta,\eta.\mu}(x,y)\prod_{a=1}^{N}\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a)\prod_{\beta=1}^{L_2}(\zeta_\alpha - y_a)}{\prod_{j=1}^{M_1}(\eta_j - x_a)\prod_{k=1}^{M_2}(\mu_k - y_a)}d\mu(x,y), \tag{1.19}$$

together with their Hilbert transforms $\{\tilde{P}_n^{(\xi,\zeta,\eta,\mu)}(x), \tilde{S}_n^{(\xi,\zeta,\eta,\mu)}(y)\}_{n\in\mathbf{N}}$ may immediately be deduced from formula (1.9), simply by making the respective replacements

$$\begin{aligned}
&\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a)\rightarrow(x-x_a)\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a), &&\prod_{\beta=1}^{L_2}(\zeta_\beta - y_a)\rightarrow(y-y_a)\prod_{\beta=1}^{L_2}(\zeta_\beta - y_a),\\
&\prod_{j=1}^{M_1}(\eta_j - x_a)\rightarrow(x-x_a)\prod_{j=1}^{M_1}(\eta_j - x_a), &&\prod_{k=1}^{M_2}(\mu_k - x_a)\rightarrow(y-y_a)\prod_{k=1}^{M_2}(\mu_k - x_a)
\end{aligned} \tag{1.20}$$

in $\mathbf{I}_n^{(2)}$ and multiplying the result by the normalizing factor $\sqrt{\frac{\mathbf{I}_{n+1}^{(2)}}{(n+1)\mathbf{I}_n^{(2)}}}$.

Remark 1.3 The corresponding formulae for the cases where one or more of the parameters within the groups $\{\xi_\alpha\}$, $\{\zeta_\beta\}$, $\{\eta_j\}$, $\{\mu_k\}$ coincide may be very easily determined from (1.9), simply by taking the appropriate limits. This replaces the terms which appear multiply in the entries in (1.11) and the sums (1.12) - (1.15) by their derivatives with respect to the repeated parameters.

As a "warm-up exercise", we also rederive the simpler analogous results for integrals of symmetric rational functions in one set of N variables arising, e.g., in Hermitian one-matrix models. In this case, let $\{P_n(x)\}_{n=0,1\dots}$ denote the orthogonal polynomials with respect to a measure $d\mu(x)$, which may again, in general, be complex and supported on an arbitrary finite union of curve segments:

$$\int d\mu(x)P_n(x)P_m(x) = \delta_{nm}, \tag{1.21}$$

with leading term normalization

$$P_n(x) = \frac{x^n}{\sqrt{h_n}} + O(x^{n-1}). \tag{1.22}$$

To guarantee their existence, the finiteness of the Hankel matrix of moments

$$M_{jk} := \int d\mu(x)x^{j+k} < \infty, \quad 0, \quad \forall j,k\in\mathbf{N}. \tag{1.23}$$

is assumed, as well as the nonsingularity of all diagonal $N\times N$ submatrices $(M_{jk})_{0\le j,k\le N-1}$. The Hilbert transforms

$$\tilde{P}_n(\eta) := \int d\mu(x)\frac{P_n(x)}{\eta - x} \tag{1.24}$$

of the orthogonal polynomials are again assumed to exist. The partition function is

$$\mathbf{Z}_N := \int d(\mu)x_1)\dots\int d\mu(x_N)\Delta_N^2(x) = N!\prod_{n=0}^{N-1} h_n. \tag{1.25}$$

The expression analogous to (1.9) for the integral of a symmetric, rational function in the N variables $\{x_a\}_{a=1,\dots N}$, having simple zeros at the points $\{\xi_\alpha\}_{\alpha=1\dots L}$, and simple poles at $\{\eta_j\}_{j=1\dots M}$, valid for $N\ge M$, is

$$\mathbf{I}_N := \frac{1}{\mathbf{Z}_N}\int d\mu(x_1)\dots\int d\mu(x_N)\prod_{a=1}^{N}\frac{\prod_{\alpha=1}^{L}(\xi_\alpha - x_a)}{\prod_{j=1}^{M}(\eta_j - x_a)}\Delta_N^2(x) \tag{1.26}$$

$$= \frac{(-1)^{\frac{M(M-1)}{2}+LM}\prod_{n=N}^{N+L-1}\sqrt{h_n}}{\Delta_L(\xi)\Delta_M(\eta)\prod_{n=N-M}^{N-1}\sqrt{h_n}}\det\begin{pmatrix} P_{N-M}(\xi_\alpha) & \dots & P_{N+L-1}(\xi_\alpha) \\ \tilde{P}_{N-M}(\eta_j) & \dots & \tilde{P}_{N+L-1}(\eta_j)\end{pmatrix}, \tag{1.27}$$

where the two groups of complex parameters $\{\xi_\alpha\}_{\alpha=1,\dots L}$, $\{\eta_j\}_{j=1,\dots M}$ are assumed to have distinct values, the latter being evaluated off the contours of integration. The analogous formula for the same integral, valid for $N<M$ is

$$\mathbf{I}_N = \frac{(-1)^{\frac{1}{2}N(N-1)}(-1)^{LM}\prod_{n=0}^{M-N}\sqrt{h_n}\prod_{n=N}^{N+L-1}\sqrt{h_n}}{\Delta_L(\xi)\Delta_M(\eta)\prod_{n=0}^{N-1}\sqrt{h_n}}\times$$
$$\times\det\begin{pmatrix} P_0(\xi_\alpha) & \dots & P_{N+L-1}(\xi_\alpha) & 0 & \dots & 0 \\ \tilde{P}_0(\eta_j) & \dots & \tilde{P}_{N+L-1}(\eta_j) & P_0(\eta_j) & \dots & P_{M-N}(\eta_j)\end{pmatrix}. \tag{1.28}$$

Such relations for polynomial integrands originated in the work of Heine and Christoffel [10, 14] and were extended to the general rational case in [15]. The case (1.27) was recently rederived in the context of Hermitian matrix models by other methods [5, 6, 8]. The direct method given here leads to relations (1.27), (1.28) in just a few lines.

The key tool that is used in our "direct" approach is the following identity, which is just a multivariable partial fraction expansion for rational symmetric functions, valid if $N\ge M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^{N}\prod_{j=1}^{M}(\eta_j - x_a)} = (-1)^{MN}\sum_{\sigma\in S_M}\mathrm{sgn}(\sigma)\sum_{a_1<\dots<a_M}^{N}(-1)^{\sum_{j=1}^{M}a_j}\frac{\Delta_{N-M}(x[\mathbf{a}])}{\prod_{j=1}^{M}(\eta_{\sigma_j} - x_{a_j})}. \tag{1.29}$$

Here $x[\mathbf{a}]$ denotes the sequence $(x_1,\dots x_N)$ with the elements $(x_{a_1},\dots x_{a_M})$ omitted, and $\mathrm{sgn}(\sigma)$ is the sign of the permutation $\sigma=\begin{pmatrix}1 & \dots & M\\ \sigma_1 & \dots & \sigma_M\end{pmatrix}\in S_M$. (To verify this identity, one simply notes that, viewed as rational functions in the η_j's, the residues at all poles coincide, and both sides tend to 0 as $\eta_j\to\infty$ for any j.) Reversing the rôles of $\{x_a\}$ and $\{\eta_j\}$, an equivalent identity of slightly different form is valid, in this case for $N\le M$:

$$\frac{\Delta_N(x)\Delta_M(\eta)}{\prod_{a=1}^N\prod_{j=1}^M(\eta_j-x_a)}=\frac{(-1)^{\frac{1}{2}N(N-1)}}{(M-N)!}\sum_{\sigma\in S_M}\mathrm{sgn}(\sigma)\frac{\Delta_{M-N}(\eta_{\sigma_{N+1}},\dots,\eta_{\sigma_M})}{\prod_{a=1}^N(\eta_{\sigma_a}-x_a)},\tag{1.30}$$

Another relation that is of importance is the Cauchy-Binet identity from multilinear algebra. In invariant form, this states that if V is an oriented Euclidean vector space with volume form Ω, and we have two sets of L vectors $(P^1,\dots,P^L)$, $(S^1,\dots,S^L)$, then the scalar product of their exterior products $(\wedge_{\alpha=1}^L P^\alpha,\wedge_{\beta=1}^L S^\beta)$, defined by

$$\wedge_{\alpha=1}^L P^\alpha * \wedge_{\beta=1}^L S^\beta=(\wedge_{\alpha=1}^L P^\alpha,\wedge_{\beta=1}^L S^\beta)\Omega\tag{1.31}$$

(where $*$ denotes Hodge dual on exterior forms) is equal to the determinant of the matrix formed from the scalar products

$$(\wedge_{\alpha=1}^L P^\alpha,\wedge_{\beta=1}^L S^\beta)=\mathrm{det}G\tag{1.32}$$

$$G^{\alpha\beta}:=(P^\alpha,S^\beta),\quad 1\le i,j\le L\tag{1.33}$$

In component form, if $\dim V=N+L$, and the vectors $\{P^\alpha,S^\alpha\}_{1\le\alpha\le L}$ have components $\{P_j^\alpha,S_k^\alpha\}_{1\le j,k\le L+N}$ relative to a positively oriented orthonormal frame, this reads:

$$\epsilon^{j_1\dots j_N j_{N+1}\dots j_{N+L}}\epsilon^{j_1\dots j_N k_{N+1}\dots k_{N+L}}P^1_{j_{N+1}}\dots P^L_{j_{N+L}}S^1_{k_{N+1}}\dots S^L_{k_{N+L}}=N!\mathrm{det}(G),\tag{1.34}$$

where ϵ denotes the Levi-Civita symbol and paired indices are again summed over.

It is worthwhile noting that both (1.29) and (1.31) are particular forms of determinantal identities that may be deduced from Wick's theorem for products of free Fermi field operators. In a sequel to this work [12], another method of deriving the relations (1.9), (1.27) and (1.28) is given, based directly on evaluation of vacuum state matrix elements of operators constructed from products and exponentials of Fermionic free fields.

2 The one matrix case. Proof of eqs. (1.27), (1.28)

To introduce the "direct" method used here, we begin by deriving the relation (1.27). First, we recall the proof for the case $M = 0$ given in [6].

$$
\begin{aligned}
\mathbf{I}_N &= \frac{1}{\mathbf{Z}_N} \int d\mu(x_1) \dots \int d\mu(x_N) \prod_{a=1}^{N} \prod_{\alpha=1}^{L} (\xi_\alpha - x_a) \Delta_N^2(x) \\
&= \frac{1}{N! (\prod_{n=0}^{N-1} h_n) \Delta_L(\xi)} \int d\mu(x_1) \dots \int d\mu(x_N) \Delta_{N+L}(x,\xi) \Delta_N(x) \\
&= \frac{\prod_{n=N}^{N+L-1} \sqrt{h_n}}{N! \Delta_L(\xi)} \int d\mu(x_1) \dots \int d\mu(x_N) \det\left(P_j(x_a) \quad P_j(\xi_\alpha) \right) \det(P_k(x_b)) \\
&= \frac{\prod_{n=N}^{N+L-1} \sqrt{h_n}}{N! \Delta_L(\xi)} \int d\mu(x_1) \dots \int d\mu(x_N) \epsilon^{j_1 \dots j_N j_{N+1} \dots j_{N+L}} P_{j_1}(x_1) \dots P_{j_N}(x_N) \\
&\qquad \times P_{j_{N+1}}(\xi_1) \dots P_{j_{N+L}}(\xi_L) \epsilon^{k_1 \dots k_N} P_{k_1}(x_1) \dots P_{k_N}(x_N).
\end{aligned}
\tag{2.1}
$$

Here the summation convention is used and $\epsilon^{j_1 \dots j_N j_{N+1} \dots j_{N+L}}$ and $\epsilon^{k_1 \dots k_N}$ denote the Levi-Civita symbol in $N + L$ and N variables, respectively, the ranges of summation being $0 \le j, j_1, \dots, j_{N+L} \le N + L - 1$ and $0 \le k, k_1, \dots, k_L \le N - 1$. Using the orthogonality relations (1.21) to evaluate the integrals yields

$$
\mathbf{I}_N = \frac{\prod_{n=N}^{N+L-1} \sqrt{h_n}}{\Delta_L(\xi)} \det(P_{N+\alpha-1}(\xi_\beta))_{1 \le \alpha,\beta \le L}. \tag{2.2}
$$

To extend this result to the case of arbitrary M, we make use of the identity (1.29). Substituting this into the integrand of (1.27), using symmetry under permutations of the x_a's, and invariance under relabeling of the integration variables gives

$$
\begin{aligned}
\mathbf{I}_N = \frac{(-1)^{\frac{M(M-1)}{2}} N!}{\mathbf{Z}_N \Delta_M(\eta) (N-M)!} &\int \frac{d\mu(z_1)}{\eta_1 - z_1} \dots \int \frac{d\mu(z_M)}{\eta_j - z_j} \Delta_M(z) \prod_{\alpha=1}^{L} \prod_{j=1}^{M} (\xi_\alpha - z_j) \\
\times \int d\mu(x_1) \dots \int d\mu(x_{N-M}) \Delta_{N-M}^2(x_1, \dots, x_{N-M}) &\prod_{a=1}^{N-M} \prod_{\alpha=1}^{L} (\xi_\alpha - x_a) \prod_{j=1}^{M} (z_j - x_a),
\end{aligned}
\tag{2.3}
$$

where the combinatorial factor $\frac{N!}{(N-M)!}$ has been introduced to replace the sum over all permutations of ordered choices of M of the elements $\{x_a\}_{a=1,\dots,N}$, which are here relabeled $(z_1, \dots, z_M)$, while the remaining ones are relabelled $(x_1, \dots, x_{N-M})$. Now, applying the relation (2.2) with the L parameters $\xi := (\xi_1, \dots, \xi_L)$ extended to $L + M$ parameters

$(\xi, z) := (\xi_1, \ldots, \xi_L, z_1, \ldots, z_M)$ to (2.3), and N replaced by $N - M$, and using (1.25) for both $\mathbf{Z}_N$ and $\mathbf{Z}_{N-M}$ gives

$$\mathbf{I}_N = \frac{(-1)^{\frac{M(M-1)}{2}}(-1)^{LM}\prod_{m=N-M}^{N+L-1}\sqrt{h_m}}{\Delta_L(\xi)\Delta_M(\eta)\prod_{n=N-M}^{N-1}h_n}\int\frac{d\mu(z_1)}{\eta_1 - z_1}\cdots\int\frac{d\mu(z_M)}{\eta_j - z_j}$$
$$\times \det\begin{pmatrix} P_{N-M}(\xi_\alpha) & \ldots & P_{N+L-1}(\xi_\alpha) \\ P_{N-M}(z_j) & \ldots & P_{N+L-1}(z_j)\end{pmatrix}. \tag{2.4}$$

The multilinearity of the determinant when evaluating the integrals then gives (1.27).

Now consider the case $N < M$. Substituting (1.30) into the definition (1.26) of $\mathbf{I}_N$ and using (1.25) gives

$$\begin{aligned}
\mathbf{I}_N &= \frac{(-1)^{\frac{1}{2}N(N-1)}}{N!(M-N)!\Delta_L(\xi)\Delta_M(\eta)} \\
&\quad\times\sum_{\sigma\in S_M}\int d\mu(x_1)\ldots\int d\mu(x_N)\frac{\Delta_{N+L}(x,\xi)\Delta_{M-N}(\eta_{\sigma_{N+1}},\ldots,\eta_{\sigma_M})}{\prod_{a=1}^{N}(\eta_{\sigma_a} - x_a)} \\
&= \frac{(-1)^{\frac{1}{2}N(N-1)}\prod_{n=0}^{M-N}\sqrt{h_n}\prod_{n=N}^{N+L-1}\sqrt{h_n}}{N!(M-N)!\prod_{n=0}^{N-1}\sqrt{h_n}\Delta_L(\xi)\Delta_M(\eta)}\det\left(P_j(\eta_{\sigma_{N+k}})_{\substack{0\le j\le M-N-1\\ 1\le k\le M-N}}\right) \\
&\quad\times\sum_{\sigma\in S_M}\int\frac{d\mu(x_1)}{(\eta_{\sigma_1} - x_1)}\cdots\int\frac{d\mu(x_N)}{(\eta_{\sigma_N} - x_N)}\det\begin{pmatrix} P_l(x_a)|_{\substack{0\le l\le N+L-1\\ 1\le a\le N}} \\ P_l(\xi_\alpha)|_{\substack{0\le l\le N+L-1\\ 1\le \alpha\le L}}\end{pmatrix} \\
&= \frac{(-1)^{\frac{1}{2}N(N-1)}\prod_{n=0}^{M-N}\sqrt{h_n}\prod_{n=N}^{N+L-1}\sqrt{h_n}}{N!(M-N)!\prod_{n=0}^{N-1}\sqrt{h_n}\Delta_L(\xi)\Delta_M(\eta)} \\
&\quad\times\sum_{\sigma\in S_M}\det\left(P_j(\eta_{\sigma_{N+k}})|_{\substack{0\le j\le M-N-1\\ 1\le k\le M-N}}\right)\det\begin{pmatrix} \tilde{P}_l(\eta_{\sigma_a})|_{\substack{0\le l\le N+L-1\\ 1\le a\le N}} \\ P_l(\xi_\alpha)|_{\substack{0\le l\le N+L-1\\ 1\le \alpha\le L}}\end{pmatrix},
\end{aligned}\tag{2.5}$$

which is just the block matrix determinantal expansion of (1.27).

3 The coupled matrix case

3.1 Case 1. $N + L_1 - M_1 \geq N + L_2 - M_2 \geq 0$

We now turn to the derivation of relation (1.9), which will be done along similar lines to the above. We again begin with the case with no denominator factors; i.e. $M_1 = M_2 = 0$, and $L_1 \geq L_2$. For this case, using (1.5),

$$\mathbf{I}_N^{(2)} := \frac{1}{\mathbf{Z}_N^{(2)}}\prod_{a=1}^{N}\left(\int d\mu(x_a, y_a)\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a)\prod_{\beta=1}^{L_2}(\zeta_\alpha - y_a)\right)\Delta_N(x)\Delta_N(y)$$

$$
\begin{aligned}
&= \frac{1}{N!(\prod_{n=0}^{N-1} h_n)\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)} \prod_{a=1}^{N} \left(\int d\mu(x_a, y_a) \right) \Delta_{N+L_1}(x,\xi)\Delta_{N+L_2}(y,\zeta) \\
&= \frac{\prod_{n=N}^{N+L_2-1} \sqrt{h_n} \prod_{n=N}^{N+L_1-1} \sqrt{h_n}}{N!\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)} \\
&\quad \times \prod_{a=1}^{N} \left(\int d\mu(x_a, y_a) \right) \det(P_j(x_b)\ P_j(\xi_\alpha)) \det(S_k(y_c)\ S_k(\zeta_\beta)) \\
&= \frac{\prod_{n=N}^{N+L_2-1} \sqrt{h_n} \prod_{n=N}^{N+L_1-1} \sqrt{h_n}}{N!\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)} \\
&\quad \times \epsilon^{j_1 \dots j_N j_{N+1} \dots j_{N+L_1}} P_{j_1}(x_1) \dots P_{j_N}(x_N) P_{j_{N+1}}(\xi_1) \dots P_{j_{N+L_1}}(\xi_{L_1}) \\
&\quad \times \epsilon^{j_1 \dots j_N k_{N+1} \dots k_{N+L_2}} S_{j_1}(y_1) \dots S_{j_N}(y_N) S_{k_{N+1}}(\zeta_1) \dots S_{k_{N+L_2}}(\zeta_{L_2}),
\end{aligned} \tag{3.1}
$$

where in the first determinant $0 \le j, j_1, \dots, j_{N+L_1} \le N + L_1 - 1$, and in the second $0 \le k, k_{N+1}, \dots, k_{N+L_2} \le N + L_2 - 1$. To complete the computation, we now make use of the Cauchy-Binet identity (1.34). To apply this to the expression (3.1), let $L := L_1$ and identify the vectors $\{P^\alpha, S^\beta\}_{1 \le \alpha, \beta \le L}$ as follows:

$$
\begin{aligned}
P^\alpha_j &:= P_{j-1}(\xi_\alpha), && 1 \le i \le L_1, && 0 \le j \le N + L_1 - 1, \\
S^\beta_j &:= S_{j-1}(\zeta_\beta), && 1 \le \beta \le L_2, && 0 \le j \le N + L_2 - 1, \\
S^\beta_j &:= 0, && 1 \le \beta \le L_2, && N + L_2 \le j \le N + L_1 - 1, \\
S^\beta_j &:= \delta_{N+L_2+\beta-1,j}, && L_2 + 1 \le \beta \le L_1, && 0 \le j \le N + L_1 - 1.
\end{aligned} \tag{3.2}
$$

Using the Cauchy-Binet identity (1.34) and eq. (1.5), the equality (3.1) gives the following expression for $\mathbf{I}_N^{(2)}$.

$$
\mathbf{I}_N^{(2)} = \frac{\prod_{n=N}^{N+L_2-1} \sqrt{h_n} \prod_{n=N}^{N+L_1-1} \sqrt{h_n}}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)} \det \left(\overset{N+L_2}{K_{12}}(\xi_\alpha, \zeta_\beta) \quad P_{N+L_2}(\xi_\alpha) \dots P_{N+L_1-1}(\xi_\alpha) \right), \tag{3.3}
$$

where

$$
\overset{N+L_2}{K_{12}}(\xi, \zeta) := \sum_{n=0}^{N+L_2-1} P_n(\xi) S_n(\zeta). \tag{3.4}
$$

We now extend this result to the case of arbitrary (L_1, L_2, M_1, M_2) satisfying (1.7). We detail the derivation only in the case when the stronger inequality

$$
N \ge \max(M_1,\ M_2) \tag{3.5}
$$

holds. For the intermediate cases, when N lies between $M_1 - L_1$ and M_1 or between $M_2 - L_2$ and M_2, formula (1.9) may be derived by similar computations.

Substituting the identity (1.29) for both denominator factors $\prod_{a=1}^{N}\prod_{j=1}^{M_1}(\eta_j - x_a)$ and $\prod_{a=1}^{N}\prod_{k=1}^{M_2}(\mu_k - y_a)$ into the integral (1.8) defining $\mathbf{I}_N^{(2)}$ gives:

$$\mathbf{I}^{(2)} = \frac{(-1)^{(M_1+M_2)N}}{\mathbf{Z}_N^{(2)}\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} \sum_{\sigma\in S_{M_1}} \mathrm{sgn}(\sigma) \sum_{\tilde{\sigma}\in S_{M_2}} \mathrm{sgn}(\tilde{\sigma}) \sum_{a_1<\ldots<a_{M_1}}^{N} (-1)^{\sum_{j=1}^{M_1} a_j} \sum_{b_1<\ldots<b_{M_2}}^{N} (-1)^{\sum_{k=1}^{M_2} b_k}$$
$$\times \prod_{a=1}^{N}\left(\int d\mu(x_a,y_a)\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a)\prod_{\beta=1}^{L_2}(\zeta_\beta - y_a)\right)\frac{\Delta_{N-M_1}(x[\mathbf{a}])\Delta_{N-M_2}(y[\mathbf{b}])}{\prod_{j=1}^{M_1}(\eta_{\sigma_j} - x_{a_j})\prod_{k=1}^{M_2}(\mu_{\tilde{\sigma}_k} - y_{b_k})}. \tag{3.6}$$

In this sum we must distinguish:

$m :=$ the number of a_j's that coincide with b_k's

$M_1 - m =$ the number of a_j's that do not coincide with any b_k's

$M_2 - m =$ the number of b_k's that do not coincide with any a_j's

Note that, if $M_1 + M_2 \leq N$, m can vary from 0 to $\min(M_1, M_2)$, but if $N < M_1 + M_2$, it can only take values $m \geq M_1 + M_2 - N$. The number of distinct ways in which two such ordered sets $a_1 < \ldots < a_{M_1}$, and $b_1 < \ldots < b_{M_1}$ with exactly m common elements can be chosen from the numbers $(1, \ldots, N)$ is:

$$C^N_{m,M_1,M_2} := \frac{N!}{(N - M_1 - M_2 + m)!(M_1 - m)!(M_2 - m)!m!}. \tag{3.7}$$

In view of the invariance of the integrand in (1.9) under permutations of the pairs $\{(x_a, y_a)\}_{a=1,\ldots N}$, and the freedom to relabel the integration variables, we may express the integral as

$$\mathbf{I}^{(2)} = \frac{(-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}}{\mathbf{Z}_N^{(2)}\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} \sum_{m=\max(0,M_1+M_2-N)}^{\min(M_1,M_2)} (-1)^m \sum_{\sigma\in S_{M_1}} \mathrm{sgn}(\sigma) \sum_{\tilde{\sigma}\in S_{M_2}} \mathrm{sgn}(\tilde{\sigma})$$
$$\times C^N_{m,M_1,M_2}\prod_{i=1}^{m}\left(\int d\mu(z_i,w_i)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - z_i)\prod_{\beta=1}^{L_2}(\zeta_\beta - w_i)}{(\eta_{\sigma_i} - z_i)(\mu_{\tilde{\sigma}_i} - w_i)}\right)$$
$$\times \prod_{j=1}^{M_1-m}\left(\int d\mu(\tilde{z}_j,\tilde{w}_j)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - \tilde{z}_j)\prod_{\beta=1}^{L_2}(\zeta_\beta - \tilde{w}_j)}{\eta_{\sigma_{m+j}} - \tilde{z}_j}\right)\Delta_{M_1-m}(\tilde{w})$$
$$\times \prod_{k=1}^{M_2-m}\left(\int d\mu(\tilde{\tilde{z}}_k,\tilde{\tilde{w}}_k)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - \tilde{\tilde{z}}_k)\prod_{\beta=1}^{L_2}(\zeta_\beta - \tilde{\tilde{w}}_k)}{\mu_{\tilde{\sigma}_{M_1+k}} - \tilde{\tilde{w}}_k}\right)\Delta_{M_2-m}(\tilde{\tilde{z}})$$

$$\times \prod_{a=1}^{N-M_1-M_2+m} \left(\int d\mu(x_a,y_a) \prod_{\alpha=1}^{L_1}(\xi_\alpha - x_a) \prod_{k=1}^{M_2-m} (\tilde{\tilde{z}}_k - x_a) \prod_{\beta=1}^{L_2}(\zeta_\beta - y_a) \prod_{j=1}^{M_1-m} (\tilde{w}_j - y_a) \right),$$
$$\times \qquad \Delta_{N-M_1-M_2+m}(x)\Delta_{N-M_1-M_2+m}(y) \tag{3.8}$$

where we have made the following changes of notation in the integration variables

$$\begin{aligned}
\begin{pmatrix} x_1,\dots,x_m \\ y_1,\dots,y_m \end{pmatrix} &\rightarrow \begin{pmatrix} z_1,\dots,z_m \\ w_1,\dots,w_m \end{pmatrix} \\
\begin{pmatrix} x_{m+1},\dots,x_{M_1} \\ y_{m+1},\dots,y_{M_1} \end{pmatrix} &\rightarrow \begin{pmatrix} \tilde{z}_1,\dots,\tilde{z}_{M_1-m} \\ \tilde{w}_1,\dots,\tilde{w}_{M_1-m} \end{pmatrix} \\
\begin{pmatrix} x_{M_1+1},\dots,x_{M_1+M_2-m} \\ y_{M_1+1},\dots,y_{M_1+M_2-m} \end{pmatrix} &\rightarrow \begin{pmatrix} \tilde{\tilde{z}}_1,\dots,\tilde{\tilde{z}}_{M_2-m} \\ \tilde{\tilde{w}}_1,\dots,\tilde{\tilde{w}}_{M_2-m} \end{pmatrix} \\
\begin{pmatrix} x_{M_1+M_2-m+1},\dots,x_N \\ y_{M_1+M_2-m+1}\dots,y_N \end{pmatrix} &\rightarrow \begin{pmatrix} x_1,\dots,x_{N-M_1-M_2+m} \\ y_1\dots,y_{N-M_1-M_2+m} \end{pmatrix}.
\end{aligned} \tag{3.9}$$

In determining the sign factor in the first line of (3.8), we have replaced the sums $\sum_{j=1}^{M_1} a_j$ and $\sum_{k=1}^{M_2} b_j$ appearing in (3.6) by their values for the case $(a_1,\dots,a_{M_1}) = (1,\dots,M_1)$ and $(b_1,\dots,b_{M_2}) = (1,\dots,m,M_1+1,\dots,M_1+M_2)$. This leaves a residual factor $(-1)^{m^2} = (-1)^m$, giving an alternating sign in the sum over m.

A further simplification can be made in (3.8) by noting that, in the sums over the elements of the symmetric groups S_{M_1} and S_{M_2}, all terms in the integrations over the $\{z_i,w_i\}_{i=1\dots m}$, $\{\tilde{z}_j,\tilde{w}_j\}_{j=1\dots M_1-m}$ and $\{\tilde{\tilde{z}}_j,\tilde{\tilde{w}}_k\}_{k=1\dots M_2-m}$ variables coming from pairs of permutations $(\sigma,\tilde{\sigma})$ for which the sets $\{(\sigma_i,\tilde{\sigma}_i)\}_{i=1,\dots m}$, $\{\sigma_{m+j}\}_{k=1,\dots M_1-m}$ and $\{\tilde{\sigma}_{M_1+k}\}_{k=1,\dots M_2-m}$ are invariant contribute the same value to the sum, and there are $m!(M_1-m)!(M_2-m)!$ of these. That is, all left cosets of the subgroup $S_{m,M_1,M_2} := S_m\times S_{M_1-m}\times S_{M_2-m} \subset S_{M_1}\times S_{M_2}$ that permute separately the first m elements in both $(1,\dots,M_1)$ and $(1,\dots,M_2)$, the last M_1-m elements in $(1,\dots,M_1)$ and the last M_2-m elements in $(1,\dots,M_2)$ contribute the same term in the sum. Hence we may choose one representative $[\sigma,\tilde{\sigma}] \in (S_{M_1}\times S_{M_2})/S_{m,M_1,M_2}$ from each coset, multiplying this term by the factor $m!(M_1-m)!(M_2-m)!$.

We may also now apply relation (3.3), with the replacements $N\rightarrow N-M_1-M_2+m$, $L_1\rightarrow L_1+M_2-m$, $L_2\rightarrow L_2+M_1-m$, as well as the expression (1.5) for both $\mathbf{Z}_N^{(2)}$ and $\mathbf{Z}^{(2)}_{N-M_1-M_2+m}$ to evaluate the integrals over the $\{(x_a,y_a)\}_{a=1\dots N-M_1-M_2+m}$ variables. Using (1.5), the resulting sum becomes

$$\mathbf{I}^{(2)} = \frac{(-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}}{N!\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}$$
$$\times \sum_{m=0}^{\min(M_1,M_2)} (-1)^m \sum_{[\sigma,\tilde{\sigma}]} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tilde{\sigma}) \prod_{i=1}^{m} \left(\int d\mu(z_i,w_i) \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - z_i)\prod_{\beta=1}^{L_2}(\zeta_\beta - w_i)}{(\eta_{\sigma_i}-z_i)(\mu_{\tilde{\sigma}_i}-w_i)} \right)$$

$$
\begin{aligned}
&\times \prod_{j=1}^{M_1-m}\left(\int d\mu(\tilde{z}_j,\tilde{w}_j)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha-\tilde{z}_j)}{\eta_{\sigma_{m+j}}-\tilde{z}_j}\right)\prod_{k=1}^{M_2-m}\left(\int d\mu(\tilde{\tilde{z}}_k,\tilde{\tilde{w}}_k)\frac{\prod_{\beta=1}^{L_2}(\zeta_\beta-\tilde{\tilde{w}}_k)}{\mu_{\tilde{\sigma}_{M_1+k}}-\tilde{\tilde{w}}_k}\right)\\
&\times \det\begin{pmatrix} \overset{N+L_2-M_2}{K_{12}}(\tilde{\tilde{z}}_k,\tilde{w}_j) & \overset{N+L_2-M_2}{K_{12}}(\tilde{\tilde{z}}_k,\zeta_\beta) & P_{N+L_2-M_2}(\tilde{\tilde{z}}_k) & \dots & P_{N+L_1-M_1-1}(\tilde{\tilde{z}}_k)\\ \overset{N+L_2-M_2}{K_{12}}(\xi_\alpha,\tilde{w}_j) & \overset{N+L_2-M_2}{K_{12}}(\xi_\alpha,\zeta_\beta) & P_{N+L_2-M_2}(\xi_\alpha) & \dots & P_{N+L_1-M_1-1}(\xi_\alpha)\end{pmatrix}\\
=&\ \frac{(-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{m=N}^{N+L_1-M_1-1}\sqrt{h_n}}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\\
&\times \sum_{m=0}^{\min(M_1,M_2)}(-1)^m\sum_{[\sigma,\tilde{\sigma}]}\mathrm{sgn}(\sigma)\mathrm{sgn}(\tilde{\sigma})\prod_{i=1}^{m}\mathcal{H}(\mu_{\tilde{\sigma}_i,\eta_{\sigma_i}})\\
\times \det&\begin{pmatrix} \overset{N+L_2-M_2}{\mathcal{K}_{21}}(\mu_{\tilde{\sigma}_{M_1+k}},\eta_{\sigma_{m+j}}) & \overset{N+L_2-M_2}{\mathcal{K}_{22}}(\mu_{\tilde{\sigma}_{M_1+k}},\zeta_\beta) & \tilde{\mathcal{P}}_{N+L_2-M_2}(\mu_{\tilde{\sigma}_{M_1+k}}) & \dots & \tilde{\mathcal{P}}_{N+L_1-M_1-1}(\mu_{\tilde{\sigma}_{M_1+k}})\\ \overset{N+L_2-M_2}{\mathcal{K}_{11}}(\xi_\alpha,\eta_{\sigma_{m+j}}) & \overset{N+L_2-M_2}{K_{12}}(\xi_\alpha,\zeta_\beta) & P_{N+L_2-M_2}(\xi_\alpha) & \dots & P_{N+L_1-M_1-1}(\xi_\alpha)\end{pmatrix}
\end{aligned}
\tag{3.10}
$$

where

$$\overset{J}{\mathcal{K}}_{11}(\xi,\eta) := \int d\mu(x,y)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha-x)}{\eta-x}\ \overset{J}{K}_{12}(\xi,y) \tag{3.11}$$

$$\overset{J}{\mathcal{K}}_{22}(\mu,\zeta) := \int d\mu(x,y)\frac{\prod_{\beta=1}^{L_2}(\zeta_\beta-y)}{\mu-y}\ \overset{J}{K}_{12}(x,\zeta) \tag{3.12}$$

$$\overset{J}{\mathcal{K}}_{21}(\mu,\eta) := \int d\mu(x,w)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha-x)}{\eta-x}\int d\mu(z,y)\frac{\prod_{\beta=1}^{L_2}(\zeta_\beta-y)}{\mu-y}\ \overset{J}{K}_{12}(z,w) \tag{3.13}$$

$$\mathcal{H}(\mu,\eta) := \int d\mu(x,y)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha-x)\prod_{\beta=1}^{L_2}(\zeta_\beta-y)}{(\eta-x)(\mu-y)} \tag{3.14}$$

$$\tilde{\mathcal{P}}_n(\mu) := \int d\mu(x,y)\frac{\prod_{\beta=1}^{L_2}(\zeta_\beta-y)}{\mu-y}P_n(x). \tag{3.15}$$

Here, in the first line of eq. (3.10) we have cancelled the combinatorial factor $m!(M_1-m)!(M_2-m)!$ with the corresponding expression in the denominator of C^N_{m,M_1,M_2} and used the relation (1.5) for $\mathbf{Z}^{(2)}_{N-M_1-M_2+m}$ to cancel the further factor $(N-M_1-M_2+n)!$, leaving only the normalization factor $\prod_{n=0}^{N+L_2-M_2-1}h_n\prod_{n=N+L_2-M_2}^{N+L_1-M_1-1}\sqrt{h_n}$ in the numerator. Note also that, although the range of summation in m is from 0 to $\min(M_1,M_2)$, in the case when $N<M_1+M_2$, all terms with $0\le m<M_1+M_2-N$ vanish, because the determinant factors, which are of dimension $(L_1+M_2-m)\times(L_1+M_2-m)$, have entries

that are formed from scalar products of vectors of dimension $N+L_1-M_1 < L_1+M_2-m$, and hence have less than maximal rank.

We now note that the sum over cosets $[\sigma,\tilde{\sigma}] \in (S_{M_1}\times S_{M_2})/S_{(m,M_1,M_2)}$ in (3.10) is an expansion, as a sum over homogeneous polynomials in the terms $\{\mathcal{H}(\mu_k,\eta_j)\}_{\substack{1\le j\le M_1\\ 1\le k\le M_2}}$, of the single $(L_1+M_2)\times(L_1+M_2)$ determinant

$$\begin{aligned}
\mathbf{I}^{(2)} &= \frac{(-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} \\
&\times \det\begin{pmatrix} \overset{N+L_2-M_2}{\mathcal{K}_{21}}(\mu_k,\eta_j)-\mathcal{H}(\mu_k,\eta_j) & \overset{N+L_2-M_2}{\mathcal{K}_{22}}(\mu_k,\zeta_\beta) & \tilde{\mathcal{P}}_{N+L_2-M_2}(\mu_k) & \dots & \tilde{\mathcal{P}}_{N+L_1-M_1}(\mu_k) \\ \overset{N+L_2-M_2}{\mathcal{K}_{11}}(\xi_\alpha,\eta_j) & \overset{N+L_2-M_2}{K_{12}}(\xi_\alpha,\zeta_\beta) & P_{N+L_2-M_2}(\xi_\alpha) & \dots & P_{N+L_1-M_1}(\xi_\alpha) \end{pmatrix} \\
&= \frac{(-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}(-1)^{L_1M_2}\prod_{n=N}^{N+L_2-M_2-1}h_n\prod_{n=N+L_2-M_2}^{N+L_1-M_1-1}\sqrt{h_n}}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} \\
&\times \det\begin{pmatrix} \overset{N+L_2-M_2}{\mathcal{K}_{11}}(\xi_\alpha,\eta_j) & \overset{N+L_2-M_2}{K_{12}}(\xi_\alpha,\zeta_\beta) & P_{N+L_2-M_2}(\xi_\alpha) & \dots & P_{N+L_1-M_1}(\xi_\alpha) \\ \overset{N+L_2-M_2}{\mathcal{K}_{21}}(\mu_k,\eta_j)-\mathcal{H}(\mu_k,\eta_j) & \overset{N+L_2-M_2}{\mathcal{K}_{22}}(\mu_k,\zeta_\beta) & \tilde{\mathcal{P}}_{N+L_2-M_2}(\mu_k) & \dots & \tilde{\mathcal{P}}_{N+L_1-M_1}(\mu_k) \end{pmatrix}.
\end{aligned} \tag{3.16}$$

Notice also that, by adding linear combinations of the last $L_1-M_1-L_2+M_2$ columns of the matrix in (3.16) the terms $\{\overset{N+L_2-M_2}{\mathcal{K}_{11}}(\xi_\alpha,\eta_j),\ \overset{N+L_2-M_2}{\mathcal{K}_{21}}(\mu_k,\eta_j)\}$ may be replaced by $\{\overset{N+L_1-M_1}{\mathcal{K}_{11}}(\xi_\alpha,\eta_j),\ \overset{N+L_1-M_1}{\mathcal{K}_{21}}(\mu_k,\eta_j)\}$ to give

$$\begin{aligned}
\mathbf{I}^{(2)} &= \frac{(-1)^{\frac{1}{2}(M_1+M_2)(M_1+M_2-1)}\prod_{n=N}^{N+L_2-M_2-1}\sqrt{h_n}\prod_{n=N}^{N+L_1-M_1-1}\sqrt{h_n}}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} \\
&\times \det\begin{pmatrix} \overset{N+L_1-M_1}{\mathcal{K}_{11}}(\xi_\alpha,\eta_j) & \overset{N+L_2-M_2}{K_{12}}(\xi_\alpha,\zeta_\beta) & P_{N+L_2-M_2}(\xi_\alpha) & \dots & P_{N+L_1-M_1}(\xi_\alpha) \\ \overset{N+L_1-M_1}{\mathcal{K}_{21}}(\mu_k,\eta_j)-\mathcal{H}(\mu_k,\eta_j) & \overset{N+L_2-M_2}{\mathcal{K}_{22}}(\mu_k,\zeta_\beta) & \tilde{\mathcal{P}}_{N+L_2-M_2}(\mu_k) & \dots & \tilde{\mathcal{P}}_{N+L_1-M_1}(\mu_k) \end{pmatrix}.
\end{aligned} \tag{3.17}$$

As a final step we note that by separating the integrands in (3.11)–(3.15) into the sum of their principal parts at the poles $x=\eta$ and $y=\mu$ and polynomial parts of degrees $\le L_1-1$ in x and $\le L_2-1$ in y, and using biorthogonality (which implies that $\overset{N+L_2-M_2}{K_{12}}$ is the kernel of an integral operator projecting onto the first $N+L_2-M_2$ biorthogonal polynomials), the integrals (3.11)–(3.15), for $J=N+L_1-M_1$ or $J=N+L_2-M_2$ may be reduced, at the values $\{\xi_\alpha,\zeta_\beta,\eta_j,\mu_k\}$, to the following

$$\overset{N+L_1-M_1}{\mathcal{K}_{11}}(\xi_\alpha,\eta_j) = \prod_{\alpha=1}^{L_1}(\xi_\alpha-\eta_j)\overset{N+L_1-M_1}{K_{11}}(\xi_\alpha,\eta_j) \tag{3.18}$$

$$\overset{N+L_2-M_2}{\mathcal{K}_{22}}(\mu_k, \zeta_\beta) = \prod_{\beta=1}^{L_2}(\zeta_\beta - \mu_k)\overset{N+L_2-M_2}{K_{22}}(\mu_k, \zeta_\beta) \tag{3.19}$$

$$\overset{N+L_1-M_1}{\mathcal{K}_{21}}(\mu_k, \eta_j) - \mathcal{H}(\mu_k, \eta_j) = \prod_{\alpha=1}^{L_1}(\xi_\alpha - \eta_j)\prod_{\beta=1}^{L_2}(\zeta_\beta - \mu_k)\overset{N+L_1-M_1}{K_{21}}(\mu_k, \eta_j) \tag{3.20}$$

$$\tilde{\mathcal{P}}_n(\mu_k) = \prod_{\beta=1}^{L_2}(\zeta_\beta - \mu_k)\tilde{P}_n(\mu_k), \quad n \geq L_2. \tag{3.21}$$

Substituting these expressions into the determinant term in (3.17), factoring out the diagonal matrices $\mathrm{diag}(1, \ldots, 1, \prod_{\beta=1}^{L_2}(\zeta_\beta - \mu_1), \ldots, \prod_{\beta=1}^{L_2}(\zeta_\beta - \mu_{M_2}))$, making the replacements $\overset{N+L_1-M_1}{K_{11}}(\xi_\alpha, \eta_j)$, $\overset{N+L_1-M_1}{K_{21}}(\mu_k, \eta_j) \to \overset{N+L_2-M_2}{K_{11}}(\xi_\alpha, \eta_j)$, $\overset{N+L_2-M_2}{K_{21}}(\mu_k, \eta_j)$ in the first $L_2 + M_1$ columns of the determinant, by adding linear combinations of the last $L_1 - M_1 - L_2 + M_2$ columns and factoring our the diagonal matrices $\mathrm{diag}(\prod_{\alpha=1}^{L_1}(\xi_\alpha - \eta_1), \ldots \prod_{\alpha=1}^{L_1}(\xi_\alpha - \eta_{M_1}), 1 \ldots, 1))$ on the left and right when evaluating the determinant then gives the relation (1.9).

Remark 3.1 Although the computation was done here for the case when $N \geq M_1, N \geq M_2$, the other cases, in which $M_1 - L_1 \leq N \leq M_1$, or $M_2 - L_2 \leq N \leq M_2$, or both, may be derived similarly, leading to the same formula (1.9).

Formulae analogous to (1.28) for the cases $N + L_1 - M_1 \geq 0, N + L_2 - M_2 \leq 0$ and $N + L_1 - M_1 \leq 0, N + L_2 - M_2 \leq 0$ may similarly be deduced using the second form of the partial fraction identity (1.30) whenever the inequalities relating the degrees of the numerator and denominator polynomials in (3.6) require it. These are derived in the next two subsections.

3.2 Case 2. $N + L_1 - M_1 \geq 0 \geq N + L_2 - M_2$

In this case the integral $\mathbf{I}_N^{(2)}$ is given by the following determinantal expression:

$$\begin{aligned}\mathbf{I}_N^{(2)} = \tilde{\epsilon}(L_1, L_2, M_1, M_2)&\frac{\prod_{n=0}^{N-M_1+L_1-1}\sqrt{h_n}\prod_{n=0}^{M_2-N-L_2-1}\sqrt{h_n}}{\prod_{n=0}^{N-1} h_n} \\ &\times \frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_2}(\xi_\alpha - \eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta - \mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\det(G),\end{aligned} \tag{3.22}$$

where

$$\tilde{\epsilon}(L_1, L_2, M_1, M_2) := (-1)^{\frac{1}{2}N(N-1)+\frac{1}{2}M_1(M_1+1)+\frac{1}{2}L_2(L_2-1)+M_1N+M_2L_1+M_1L_2+L_1L_2} \tag{3.23}$$

and G is the $(L_1+M_2)\times(L_1+M_2)$ matrix

$$G := \det\begin{pmatrix} \frac{1}{\eta_j-\xi_\alpha} & 0 & P_b(\xi_\alpha) & 0 \\ H(\mu_k,\eta_j) & \frac{1}{\mu_k-\zeta_\beta} & \tilde{P}_b(\mu_k) & S_m(\mu_k) \end{pmatrix}. \tag{3.24}$$

Here the row indices are, sequentially, $1\le\alpha\le L_1$ and $1\le k\le M_2$, and the column indices $1\le j\le M_1$, $1\le\beta\le L_2$, $0\le b\le N-M_1+L_1-1$ and $0\le m\le M_2-N-L_2-1$. Note that the first three column blocks of (3.24) coincide with those of the matrix G defined in (1.11) if one understands the orthogonal polynomials $P_n(x), S_n(y)$ and their Hilbert transforms to vanish for negative n. Formula (3.22) is valid whenever $N+L_1-M_1\ge 0$, $N+L_2-M_2\le 0$, but it is easier to demonstrate assuming the stronger conditions

$$N\ge M_1, \quad N+L_2-M_2\le 0, \tag{3.25}$$

which is what we do in the following. The intermediate case, when $M_1-L_1\le N\le M_1$ may be proved through a similar computation.

Expressing $\mathbf{I}_N^{(2)}$ in this case as:

$$\begin{aligned}\mathbf{I}_N^{(2)} := \frac{1}{\mathbf{Z}_N^{(2)}}\prod_{a=1}^{N}\left(\int d\mu(x_a,y_a)\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha-x_a)}{\prod_{j=1}^{M_1}(\eta_j-x_a)\prod_{k=1}^{M_2}(\mu_k-y_a)}\right)\\ \times\frac{\Delta_N(x)\Delta_{N+L_2}(y,\zeta)}{\Delta_{L_2}(\zeta)}\end{aligned} \tag{3.26}$$

and applying identity (1.29) with respect to the (x,η) variables and (1.30) with respect to the $((y,\zeta)),\mu)$ variables gives

$$\begin{aligned}\mathbf{I}_N^{(2)} &= \frac{(-1)^{M_1N+\frac{1}{2}(N+L_2)(N+L_2-1)}\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\mu_k-\zeta_\beta)}{N!(M_2-N-L_2)!(\prod_{n=0}^{N-1}h_n)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\\ &\quad\times\sum_{a_1\le\ldots\le a_{M_1}}(-1)^{\sum_{j=1}^{M_1}a_j}\sum_{\sigma\in S_{M_1}}\mathrm{sgn}(\sigma)\sum_{\widetilde{\sigma}\in S_{M_2}}\mathrm{sgn}(\widetilde{\sigma})\frac{\Delta_{M_2-L_2-N}(\eta_{\widetilde{\sigma}_{N+L_2+1}},\ldots,\eta_{\widetilde{\sigma}_{M_2}})}{\prod_{\beta=1}^{L_2}(\mu_{\widetilde{\sigma}_{N+\beta}}-\zeta_\beta)}\\ &\quad\times\left(\prod_{a=1}^{N}\int\frac{d\mu(x_a,y_a)\prod_{\alpha=1}^{L_1}(\xi_\alpha-x_a)}{\mu_{\widetilde{\sigma}_a}-y_a}\right)\frac{\Delta_{N-M_1}(x[a])}{\prod_{j=1}^{M_1}(\eta_{\sigma_j}-x_j)}\\ &= \frac{(-1)^{M_1N+\frac{1}{2}(N+L_2)(N+L_2-1)+\frac{1}{2}M_1(M_1+1)}\prod_{k=1}^{L_2}\prod_{\beta=1}^{M_2}(\mu_k-\zeta_\beta)}{(M_2-N-L_2)!M_1!(N-M_1)!(\prod_{n=0}^{N-1}h_n)\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\\ &\quad\times\sum_{\sigma\in S_{M_1}}\mathrm{sgn}(\sigma)\sum_{\widetilde{\sigma}\in S_{M_2}}\mathrm{sgn}(\widetilde{\sigma})\prod_{j=1}^{M_1}\left(\int\frac{d\mu(z_j,w_j)\prod_{\alpha=1}^{L_2}(\xi_\alpha-z_j)}{(\eta_{\sigma_j}-z_j)(\mu_{\widetilde{\sigma}_j}-w_j)}\right)\end{aligned}$$

$$\times \prod_{a=1}^{N-M_1} \left(\int \frac{d\mu(x_a, y_a)}{\mu_{\widetilde{\sigma}_{M_1+a}} - y_a} \right) \det \begin{pmatrix} x_a^b \\ \xi_\alpha^b \end{pmatrix}_{\substack{1\le a\le N-M_1,\ 1\le\alpha\le L_1 \\ 0\le b\le N-M_1+L_1-1}}$$

$$\times \frac{\det(\mu^m_{\widetilde{\sigma}_{N+L_2+k}})_{0\le m\le M_2-N-L_2-1,\ 1\le k\le M_2-N-L_2}}{\prod_{\beta=1}^{L_2}(\mu_{\widetilde{\sigma}_{N+\beta}} - \zeta_\beta)} \tag{3.27}$$

$$= \frac{(-1)^{M_1 N+\frac{1}{2}(N+L_2)(N+L_2-1)+\frac{1}{2}M_1(M_1+1)} \prod_{k=1}^{L_2}\prod_{\beta=1}^{M_2}(\mu_k - \zeta_\beta)}{(M_2-N-L_2)!M_1!(N-M_1)!(\prod_{n=0}^{N-1} h_n)\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}$$

$$\times \sum_{\sigma\in S_{M_1}} \mathrm{sgn}(\sigma) \sum_{\widetilde{\sigma}\in S_{M_2}} \mathrm{sgn}(\widetilde{\sigma}) \prod_{j=1}^{M_1} \left(\mathcal{H}^\xi(\mu_{\widetilde{\sigma}_j}, \eta_{\sigma_j}\right) \det \begin{pmatrix} X_b(\mu_{\widetilde{\sigma}_{M_1+a}}) \\ \xi_\alpha^b \end{pmatrix}_{\substack{1\le a\le N-M_1,\ 1\le\alpha\le L_1 \\ 0\le b\le N-M_1+L_1-1}}$$

$$\times \prod_{\beta=1}^{L_2} \left(\frac{1}{\mu_{\widetilde{\sigma}_{N+\beta}} - \zeta_\beta} \right) \det(\mu^m_{\widetilde{\sigma}_{N+L_2+k}})_{0\le m\le M_2-N-L_2-1,\ 1\le k\le M_2-N-L_2}, \tag{3.28}$$

where

$$\mathcal{H}^\xi(\mu, \eta) := \int \frac{d\mu(x,y)\prod_{\alpha=1}^{L_1}(\xi_\alpha - x)}{(\eta - x)(\mu - y)} \tag{3.29}$$

$$X_b(\mu) := \int d\mu(x,y) \frac{x^b}{\mu - y}. \tag{3.30}$$

We now note that (3.28) is just the block determinant expansion of

$$\mathbf{I}_N^{(2)} = \frac{(-1)^{M_1 N+\frac{1}{2}(N+L_2)(N+L_2-1)+\frac{1}{2}M_1(M_1+1)} \prod_{k=1}^{L_2}\prod_{\beta=1}^{M_2}(\mu_k - \zeta_\beta)}{(\prod_{n=0}^{N-1} h_n)\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}$$

$$\times \det \begin{pmatrix} \mathcal{H}^\xi(\mu_k, \eta_j) & X_b(\mu_k) & \frac{1}{\mu_k - \zeta_\beta} & \mu_k^m \\ 0 & \xi_\alpha^b & 0 & 0 \end{pmatrix}_{\substack{0\le b\le N-M_1-L_1-1,\ 0\le m\le M_2-N-L_2-1 \\ 1\le\alpha\le L_1,\ 1\le\beta\le L_2,\ 1\le j\le M_1,\ 1\le k\le M_2}} \tag{3.31}$$

(The combinatorial factors $(M_2 - L - L_2)!M_1!(N - M_1)!$ in (3.28) are cancelled by the multiplicity with which the subdeterminant factors occur in the sums over $\sigma \in S_{M_1}$, $\widetilde{\sigma} \in S_{M_2}$.) A further simplification can be achieved by applying elementary column operations. To see this, we take the integrations appearing in the matrix elements of (3.31) outside the determinant, re-writing it as:

$$\mathbf{I}_N^{(2)} = \frac{(-1)^{M_1 N+\frac{1}{2}(N+L_2)(N+L_2-1)+\frac{1}{2}M_1(M_1+1)} \prod_{k=1}^{L_2}\prod_{\beta=1}^{M_2}(\mu_k - \zeta_\beta)}{(\prod_{n=0}^{N-1} h_n)\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}$$

$$\times \prod_{k=1}^{M_2} \left(\int \frac{d\mu(x_k, y_k)}{(\mu_k - y_k)} \right) \det \begin{pmatrix} \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_k)}{(\eta_j - x_k)} & x_k^b & \frac{1}{\mu_k - \zeta_\beta} & \mu_k^m \\ 0 & \xi_\alpha^b & 0 & 0 \end{pmatrix}. \tag{3.32}$$

Now, viewing $\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_k)}{(\eta_j - x_k)}$ as a rational function in x_k with a simple pole at $x_k = \eta_j$, we may re-express it as the sum of the pole term plus a polynomial of degree $\leq L_1 - 1$

$$\frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_k)}{(\eta_j - x_k)} = \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - \eta_j)}{(\eta_j - x_k)} + \sum_{\gamma=0}^{L_1-1} \Lambda_{j\gamma} x_k^\gamma, \tag{3.33}$$

where

$$\sum_{\gamma=0}^{L_1-1} \Lambda_{j\gamma} \xi_\alpha^\gamma = \prod_{\substack{\beta=1 \\ \beta\neq\alpha}}^{L_1} (\xi_\beta - \eta_j). \tag{3.34}$$

Since $N - M_1 + L_1 - 1 \geq L_1 - 1$, all monomials in x_k of degree $\leq L_1 - 1$ appear in the second column block of the determinant in (3.32). We may therefore add linear combinations of these columns to those in the first block to obtain the equivalent expression

$$\begin{aligned}
&\det\begin{pmatrix} \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - x_k)}{(\eta_j - x_k)} & x_k^b & \frac{1}{\mu_k - \zeta_\beta} & \mu_k^m \\ 0 & \xi_\alpha^b & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} \frac{\prod_{\alpha=1}^{L_1}(\xi_\alpha - \eta_j)}{(\eta_j - x_k)} & x_k^b & \frac{1}{\mu_k - \zeta_\beta} & \mu_k^m \\ -\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{L_1}(\xi_\beta - \eta_j) & \xi_\alpha^b & 0 & 0 \end{pmatrix} \\
&= \prod_{j=1}^{M_2}\prod_{\alpha=1}^{L_1}(\xi_\alpha - \eta_j) \det\begin{pmatrix} \frac{1}{\eta_j - x_k} & x_k^b & \frac{1}{\mu_k - \zeta_\beta} & \mu_k^m \\ \frac{1}{\eta_j - \xi_\alpha} & \xi_\alpha^b & 0 & 0 \end{pmatrix} \\
&= \prod_{n=0}^{N-M_1+L_1-1}\sqrt{h_n} \prod_{n=0}^{M_2-N-L_2-1}\sqrt{h_n} \prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_2}(\xi_\alpha - \eta_j) \det\begin{pmatrix} \frac{1}{\eta_j - x_k} & P_b(x_k) & \frac{1}{\mu_k - \zeta_\beta} & S_m(\mu_k) \\ \frac{1}{\eta_j - \xi_\alpha} & P_b(\xi_\alpha) & 0 & 0 \end{pmatrix},
\end{aligned} \tag{3.35}$$

where further elementary column operations were made in the last lign to replace the monomials x_k^b, ξ_α^b and μ_k^m by the corresponding biorthogonal polynomials. Substituting this expression into (3.32) and evaluating the integrals gives

$$\begin{aligned}
\mathbf{I}_N^{(2)} = &\frac{(-1)^{M_1 N + \frac{1}{2}(N+L_2)(N+L_2-1) + \frac{1}{2}M_1(M_1+1)} \prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_2}(\xi_\alpha - \eta_j) \prod_{k=1}^{L_2}\prod_{\beta=1}^{M_2}(\mu_k - \zeta_\beta)}{(\prod_{n=0}^{N-1} h_n)\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)} \\
&\times \prod_{n=0}^{N-M_1+L_1-1}\sqrt{h_n} \prod_{n=0}^{M_2-N-L_2-1}\sqrt{h_n} \det\begin{pmatrix} H(\mu_k, \eta_j) & \tilde{P}_b(\mu_k) & \frac{1}{\mu_k - \zeta_\beta} & S_m(\mu_k) \\ \frac{1}{\eta_j - \xi_\alpha} & P_b(\xi_\alpha) & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.36}$$

Finally, reordering the rows and columns suitably, bringing the third column block into the second position and interchanging the two row blocks, we arrive at the expression (3.22).

3.3 Case 3. $N+L_1-M_1\le 0$, $N+L_2-M_2\le 0$

In this case the integral $\mathbf{I}_N^{(2)}$ is given by the following determinantal expression:

$$\mathbf{I}_N^{(2)} = \frac{\prod_{n=0}^{M_1-L_1-N}\sqrt{h_n}\prod_{n=0}^{M_2-L_2-N}\sqrt{h_n}}{\prod_{n=0}^{N-1}h_n} \times \frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_2}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{M_2}(\mu)}\det(G), \tag{3.37}$$

where G is the $(M_1+M_2-N)\times(M_1+M_2-N)$ matrix

$$G := \det\begin{pmatrix} H(\mu_k,\eta_j) & S_m(\mu_k) & \frac{1}{\mu_k-\zeta_\beta} \\ P_\ell(\eta_j) & 0 & 0 \\ \frac{1}{\eta_j-\xi_\alpha} & 0 & 0 \end{pmatrix}, \tag{3.38}$$

with rows labelled consecutively by $1\le\alpha\le L_1$, $1\le k\le M_2$, and $0\le\ell\le M_1-L_1-N$ and the columns by $1\le j\le M_1$, $1\le\beta\le L_2$ and $0\le m\le M_2-L_2-N$.

To derive this formula, we begin by expressing the integral in the form

$$\mathbf{I}_N^{(2)} = \frac{\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{Z_M^{(2)}\Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)}\int d\mu(x_1,y_1)\dots\int d\mu(x_N,y_N) \times \frac{\Delta_{N+L_1}(x,\xi)\Delta_{N+L_2}(y,\eta)}{\prod_{a=1}^{N}\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)(\eta_j-x_a)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)(\mu_k-y_a)}. \tag{3.39}$$

Using the identity (1.30) twice, first with the x_a variables replaced by the combined set (x_a,ξ_j) and $N\to N+L_1$, and also for the set (y_a,ζ_k) with $N\to N+L_2$, we obtain

$$\begin{aligned}\mathbf{I}_N^{(2)} &= \frac{(-1)^{L_1(N+M_1)+L_2(N+M_2)+\frac12 L_1(L_1+1)+\frac12 L_2(L_2+1)}\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{N!(M_1-L_1-N)!(M_2-L_2-N)!(\prod_{n=0}^{N-1}h_n)\ \Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{m_2}(\mu)} \\ &\quad\times\sum_{\sigma\in S_{M_1}}\sum_{\tilde\sigma\in S_{M-2}}\frac{\Delta_{M_1-L_1-N}(\eta_{\sigma_{N+L_1+1}},\dots,\eta_{\sigma_{M_1}})\Delta_{M_2-L_2-N}(\mu_{\tilde\sigma_{N+L_2+1}},\dots,\mu_{\tilde\sigma_{M_2}})}{\prod_{\alpha=1}^{L_1}(\eta_{\sigma_{N+\alpha}}-\xi_\alpha)\prod_{\beta=1}^{L_2}(\mu_{\tilde\sigma_{N+\beta}}-\zeta_\beta)} \\ &\quad\times\prod_{a=1}^{N}\left(\int\frac{d\mu(x_a,y_a)}{(\eta_{\sigma_a}-x_a)(\mu_{\tilde\sigma_a}-y_a)}\right) \\ &= \frac{(-1)^{L_1(N+M_1)+L_2(N+M_2)+\frac12 L_1(L_1+1)+\frac12 L_2(L_2+1)}\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{N!(M_1-L_1-N)!(M_2-L_2-N)!(\prod_{n=0}^{N-1}h_n)\ \Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{m_2}(\mu)} \\ &\quad\times\sum_{\sigma\in S_{M_1}}\sum_{\tilde\sigma\in S_{M-2}}\left(\prod_{a=1}^{N}H(\mu_{\tilde\sigma_a},\eta_{\sigma_a})\right)\left(\prod_{\alpha=1}^{L_1}\frac{1}{\eta_{\sigma_{N+\alpha}}-\xi_\alpha}\right)\left(\prod_{\beta=1}^{L_2}\frac{1}{\mu_{\tilde\sigma_{N+\beta}}-\zeta_\beta}\right)\end{aligned}$$

$$
\begin{aligned}
&\times \det(\eta^\ell_{\sigma_{N+L_1+j}})_{\substack{1\le j\le M_1-L_1-N\\ 0\le \ell\le M_1-L_1-N}} \det(\mu^m_{\widetilde{\sigma}_{N+L_2+k}})_{\substack{1\le k\le M_2-L_2-N\\ 0\le m\le M_2-L_2-N}}\\
= &\frac{(-1)^{L_1(N+M_1)+L_2(N+M_2)+\frac{1}{2}L_1(L_1+1)+\frac{1}{2}L_2(L_2+1)}\prod_{\alpha=1}^{L_1}\prod_{j=1}^{M_1}(\xi_\alpha-\eta_j)\prod_{\beta=1}^{L_2}\prod_{k=1}^{M_2}(\zeta_\beta-\mu_k)}{(\prod_{n=0}^{N-1}h_n)\ \Delta_{L_1}(\xi)\Delta_{L_2}(\zeta)\Delta_{M_1}(\eta)\Delta_{m_2}(\mu)}\\
&\times \sum_{\sigma\in S_{M_1}}\sum_{\widetilde{\sigma}\in S_{M-2}}\left(\prod_{a=1}^{N} H(\mu_{\widetilde{\sigma}_a},\eta_{\sigma_a})\right)\left(\prod_{\alpha=1}^{L_1}\frac{1}{\eta_{\sigma_{N+\alpha}}-\xi_\alpha}\right)\left(\prod_{\beta=1}^{L_2}\frac{1}{\mu_{\widetilde{\sigma}_{N+\beta}}-\zeta_\beta}\right)\\
&\times \det\begin{pmatrix} H(\mu_k,\eta_j) & \frac{1}{\mu_k-\zeta_\beta} & \mu_k^m\\ \frac{1}{\eta_j-\xi_\alpha} & 0 & 0\\ \eta_j^\ell & 0 & 0\end{pmatrix},
\end{aligned}
\tag{3.40}
$$

where the rows are labelled consecutively by $1\le k\le M_2$, $1\le \alpha\le L_1$ and $0\le \ell\le M_1-L_1-N$ and the columns by $1\le j\le M_1$, $1\le \beta\le L_2$ and $0\le m\le M_2-L_2-N$. Finally, interchanging the two row blocks, and replacing the monomial entries η_j^l and μ_k^m by the biorthogonal polynomials $P_j(\eta_j)k)$ and $S_m(\mu_k)$ respectively, with suitably modified normalization factors, we arrive at the expression (3.37) for $\mathbf{I}_N^{(2)}$.

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